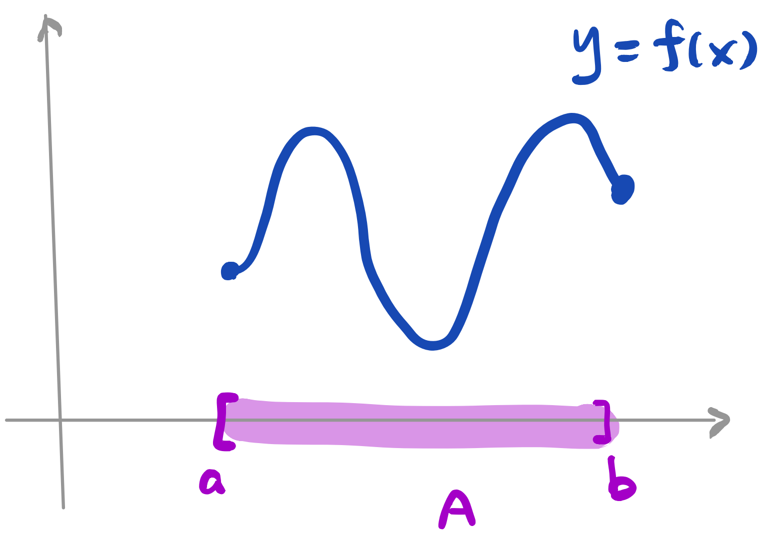
MATH 2050C Lecture 20 (Mar 29)

Last time: $f: A \to \mathbb{R}$ cts at $c \in A$ (or on $B \subseteq A$)

Seq. Criteria, construct new cts functions ......

Q: What if $f: A \to \mathbb{R}$ is a cts function defined on an closed & bdd interval $A = [a,b] \subseteq \mathbb{R}$?

Picture:



$f: [a,b] \to \mathbb{R}$ is cts (everywhere)

$\overset{\text{def}^{\underline{n}}}{\Longleftrightarrow}$ f is cts at <u>EVERY</u> $c \in [a,b]$

$\overset{\text{def}^{\underline{n}}}{\Longleftrightarrow}$ $\forall c \in [a,b]$, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, c) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \text{ when } |x - c| < \delta, \ x \in [a,b]$$

Q: What's special about $A = [a,b]$?

- All $c \in [a,b]$ ARE cluster pt of $A = [a,b]$.

So, "f cts at c" $\Leftrightarrow$ "$\lim_{x \to c} f(x) = f(c)$"

- Nested Interval Property
- Bolzano-Weierstrass Thm

} "compactness" of $A = [0,1]$

- "connectedness" of $A = [0,1]$.

We will prove 3 important theorems (§ 5.3 textbk)

(1) Boundedness Thm
(2) Extreme Value Thm

} "compactness"

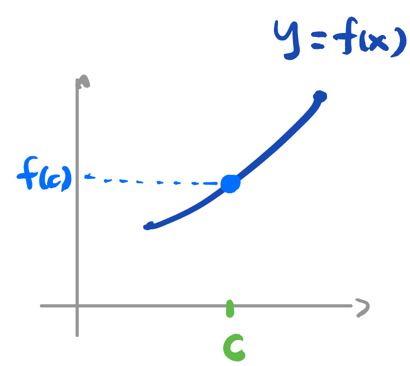
(3) Intermediate Value Thm } "connectedness"

Recall: $f: A \to \mathbb{R}$ cts at $c \in A$

$\Rightarrow$ f is "locally bdd" near c

i.e. $\exists M > 0$, $\exists \delta > 0$ s.t.

$$|f(x)| \leq M \quad \forall x \in A, |x - c| < \delta$$



Caution: M and δ depends on c in general.

Boundedness Theorem:

Any cts $f : [a,b] \to \mathbb{R}$ is bdd (globally on $[a,b]$).

i.e. $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$.

<u>Proof:</u> By contradiction, suppose in contrary that f is <u>NOT</u> bdd on $[a,b]$.

$\Rightarrow \forall n \in \mathbb{N}, \exists x_n \in [a,b]$ s.t. $|f(x_n)| > n$

① (x_n) is a seq. in $[a,b]$, hence is a bdd seq.

BWT $\Rightarrow \exists$ subseq. (x_{n_k}) of (x_n) s.t.

$$\lim_{k\to\infty} (x_{n_k}) = x_* \text{ for some } x_* \in \mathbb{R}$$

<u>Note:</u> $a \leq x_{n_k} \leq b \quad \forall k \in \mathbb{N}$

Limit Thm $\Rightarrow a \leq x_* \leq b$ ie $x_* \in [a,b]$.

② f is cts on $[a,b]$, in particular, at $x_* \in [a,b]$.

cts $\Rightarrow \lim_{x \to x_*} f(x) = f(x_*)$

Seq. Criteria $\Rightarrow \lim_{k\to\infty} f(x_{n_k}) = f(x_*)$

So, $(f(x_{n_k}))_{k \in \mathbb{N}}$ is a convergent seq. thus must be bdd. However, by construction of x_n,

$$|f(x_{n_k})| > n_k \geq k \quad \forall\, k \in \mathbb{N}$$

$\Rightarrow$ $(f(x_{n_k}))$ is unbdd, which is a contradiction.

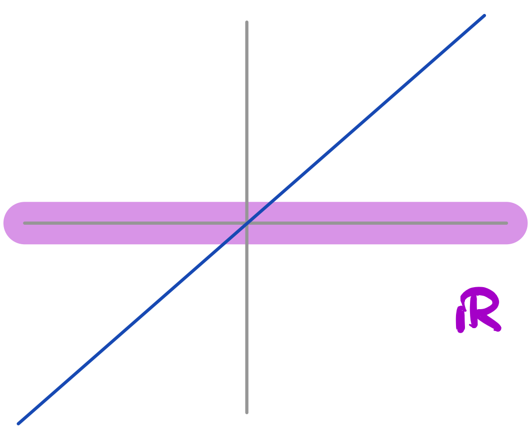
$\blacksquare$

Remark: All assumptions are required for the theorem to hold.

(1) unbdd interval.

$f : \mathbb{R} \to \mathbb{R}$

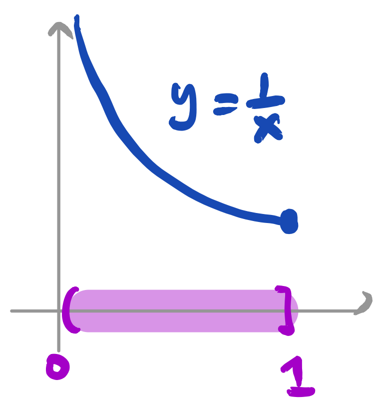
$f(x) = x$



(2) interval is NOT closed

$f : (0,1] \to \mathbb{R}$

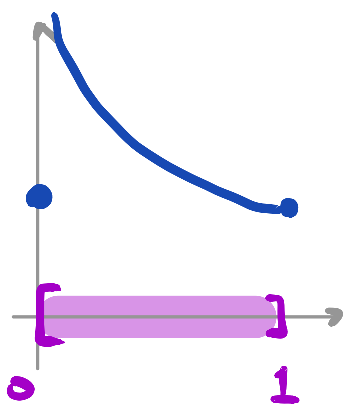
$f(x) = \frac{1}{x}$



(3) not cts

$f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$



By Boundedness Thm, $f : [a,b] \to \mathbb{R}$ cts

$\Rightarrow \quad \phi \neq \{ f(x) \mid x \in [a,b] \} \subseteq \mathbb{R}$ is bdd

By Completeness of $\mathbb{R}$, there exist

$$M := \sup \{ f(x) \mid x \in [a,b] \}$$

$$m := \inf \{ f(x) \mid x \in [a,b] \}$$

In fact, these sup & inf are achieved.

Extreme Value Theorem

A cts $f : [a,b] \to \mathbb{R}$ always achieve it's (absolute) maximum and minimum, ie

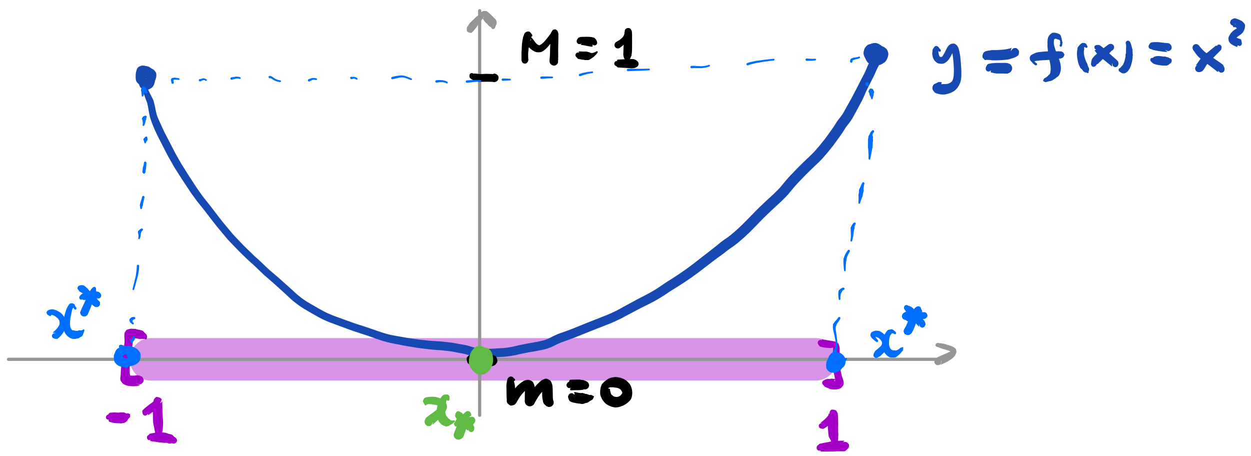
$$\exists\, x^* \in [a,b] \text{ st. } f(x^*) = M = \sup \{ f(x) \mid x \in [a,b] \}$$

$$\exists\, x_* \in [a,b] \text{ st. } f(x_*) = m = \inf \{ f(x) \mid x \in [a,b] \}$$

Remark: The theorem guarantees the "existence" of maxima x^* and minina x_*, but NOT their "uniqueness".

For example. $f : [-1, 1] \to \mathbb{R}$

$$f(x) := x^2$$

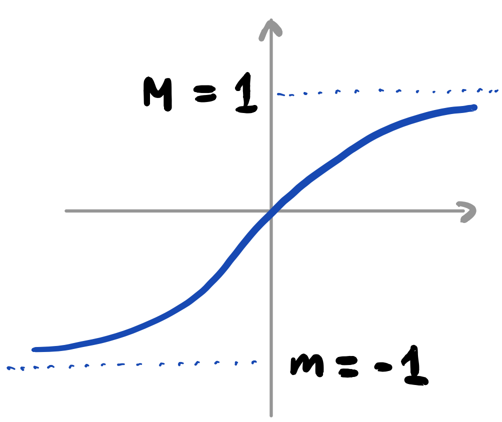


<u>Remark</u> : All assumptions are required in the theorem

(1) unbdd interval

$f : \mathbb{R} \to \mathbb{R}$

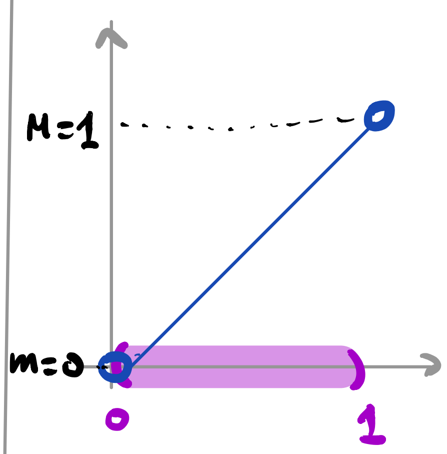
$f(x) = \tanh x$



(2) interval NOT closed.

$f : (0,1) \to \mathbb{R}$

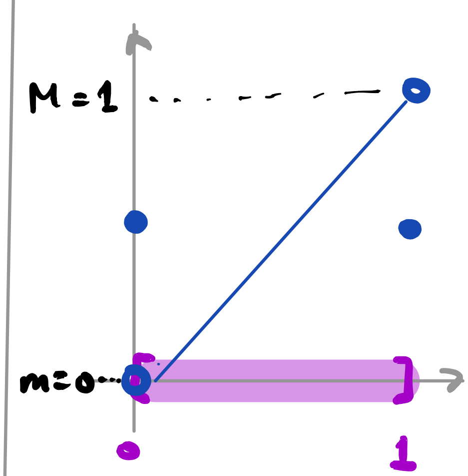
$f(x) = x$



(3) NOT cts

$f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } x \neq 0,1 \\ \frac{1}{2} & \text{if } x = 0,1 \end{cases}$$



Proof of Extreme Value Theorem:

We prove the existence of a maxima x^* and leave the minima x_* as an exercise.

Recall: $M := \sup\{f(x) \mid x \in [a,b]\}$

By def^n of supremum, $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in [a,b]$

s.t. $\quad M - \varepsilon < f(x_\varepsilon) \leq M$

Take $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, we obtain a seq.

$(x_n) \subseteq [a,b]$ s.t. $\quad M - \frac{1}{n} < f(x_n) \leq M \quad \forall n \in \mathbb{N}$

As before, BWT $\Rightarrow \exists$ subseq. of (x_n)

$$(x_{n_k}) \longrightarrow x^* \in [a,b]$$

Claim: $f(x^*) = M$, ie x^* is a maxima.

Pf: By construction.

$$M - \frac{1}{n_k} < f(x_{n_k}) \leq M \qquad \forall k \in \mathbb{N}$$

Take $k \to \infty$, by continuity of f at $x^* \in [a,b]$,

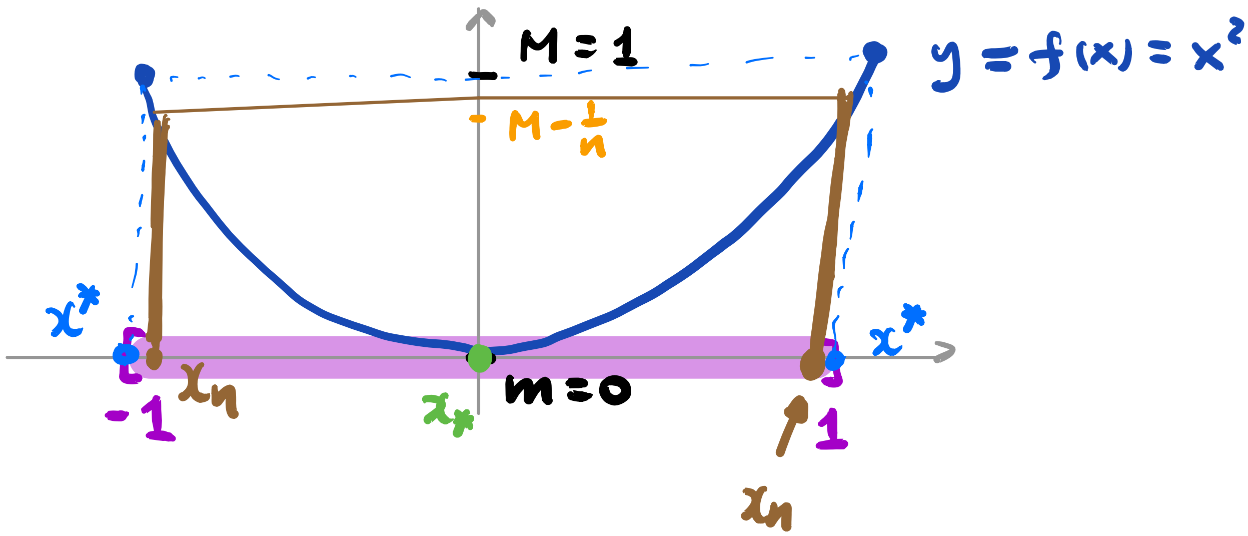
$$M \leq \lim_{k \to \infty} f(x_{n_k}) = f(x^*) \leq M$$

$\blacksquare$

Q: why non-uniqueness of maxima/minima?

For example. $f : [-1, 1] \to \mathbb{R}$

$$f(x) := x^2$$



Concerning the "connectedness" of $A = [a, b]$,

we have:

Intermediate Value Theorem

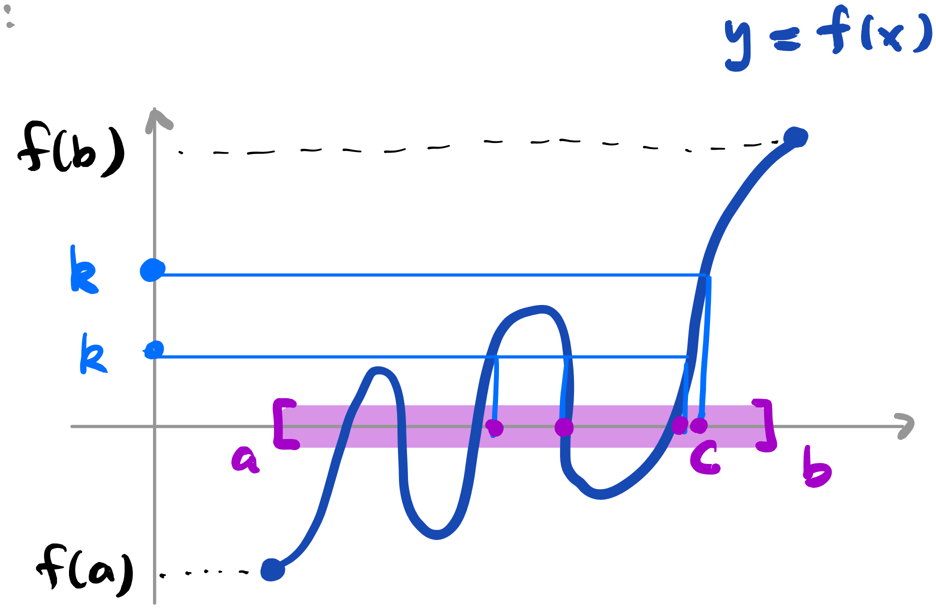
Let $f : [a, b] \to \mathbb{R}$ be a cts function s.t.

$$f(a) < f(b)$$

THEN: $\forall\, k \in (f(a), f(b))$, $\exists\, c \in [a, b]$ s.t.

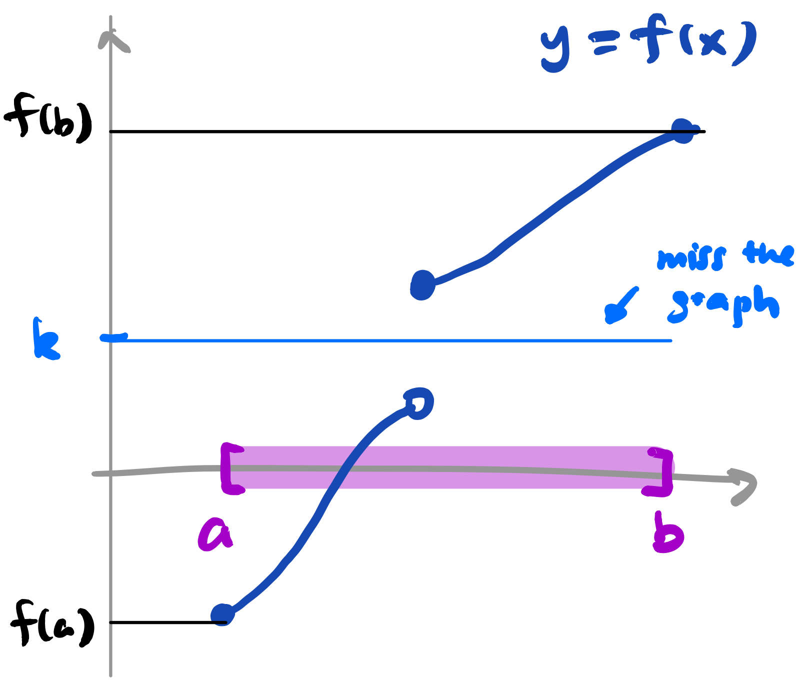
$$f(c) = k$$

Picture :



Remark : The assumptions in the theorem are all necessary.

① NOT cts



② interval NOT connected.